

# WEAKLY-EXCEPTIONAL QUOTIENT SINGULARITIES IN PRIME DIMENSION

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**ABSTRACT.** A singularity is said to be weakly-exceptional if it has a unique purely log terminal blow up. This is a natural generalization of the surface singularities of types  $D_n$ ,  $E_6$ ,  $E_7$  and  $E_8$ . Since this idea was introduced, quotient singularities of this type have been classified in dimensions up to 5. This paper looks at such singularities in dimension  $p$ , where  $p$  is an arbitrary prime number.

## 1. INTRODUCTION

The central notion of this paper was first mentioned in V. Shokurov's paper on flips (see [12]), where (among other things) he looked at the properties of exceptional divisors that appear during (partial) blow-ups of the 2-dimensional  $A$ - $D$ - $E$  singularities. This analysis has since been generalised as follows:

**Definition 1.1.** *Let  $(V \ni O)$  be a germ of a Kawamata log terminal singularity. The singularity is said to be exceptional if for every effective  $\mathbb{Q}$ -divisor  $D_V$  on the variety  $V$ , such that the log pair  $(V, D_V)$  is log canonical, there exists at most one exceptional divisor over the point  $O$  with discrepancy 1 with respect to the pair  $(V, D_V)$ .*

**Theorem 1.2** ([1, Theorem 3.7]). *Let  $(V \ni O)$  be a germ of a Kawamata log terminal singularity. Then there exists a birational morphism  $\pi : W \rightarrow V$ , such that the following hypotheses are satisfied:*

- *the exceptional locus of  $\pi$  consists of one irreducible divisor  $E$  such that  $O \in \pi(E)$ ,*
- *the log pair  $(W, E)$  has purely log terminal singularities.*
- *the divisor  $E$  is a  $\pi$ -ample  $\mathbb{Q}$ -Cartier divisor.*

*Such a morphism is called a plt blow-up of the singularity.*

**Definition 1.3.** *We say that a singularity  $(V \ni O)$  is weakly-exceptional if it has a unique plt blow-up.*

This paper seeks to extend the following example:

**Example 1.4** (see [12, Section 5.2.3]). *Consider the case of  $N = 2$ . The singularities in this case follow the well-known  $A$ - $D$ - $E$  classification, all of them being quotient singularities of the type  $\mathbb{C}^2/G$  (for different groups  $G \subset SL_2(\mathbb{C})$ ). The singularities of type  $A_n$  correspond to (reducible) lifts of cyclic groups  $\mathbb{Z}_{n+1} \subset \text{Aut}(\mathbb{P}^1)$  to  $SL_2(\mathbb{C})$ , singularities of type  $D_n$  correspond to lifts of dihedral groups  $\mathbb{D}_{n-2} \subset \text{Aut}(\mathbb{P}^1)$ , and the singularities  $E_6$ ,  $E_7$ , and  $E_8$  — to lifts of  $\mathbb{A}_4$ ,  $\mathbb{S}_4$  and  $\mathbb{A}_5$  respectively. Note that in this case the group action is irreducible exactly when the singularity is of type  $D$  or  $E$ .*

Rephrasing [12, Section 5.2.3], a singularity is exceptional exactly when it is of type  $E_6$ ,  $E_7$  or  $E_8$ ; it is weakly-exceptional exactly if it is of type  $D_n$  or  $E_n$ . Note that here a singularity is weakly-exceptional if and only if the group action that gives rise to it is irreducible (and it is exceptional if and only if the action is primitive).

The last observation turns out to be partially true in general:

**Theorem 1.5.** [8, Proposition 2.1] *If  $G \subset SL_N(\mathbb{C})$  gives rise to an exceptional singularity, then the action of  $G$  is primitive.*

**Proposition 1.6.** *If  $G \subset SL_N(\mathbb{C})$  gives rise to an weakly-exceptional singularity, then the action of  $G$  is irreducible.*

*Proof:* Similar to that of [8, Proposition 2.1]. □

**Remark 1.7.** *However, the reverses of the last two statements are not true: for instance, the group  $A_5 \subset SL_3(\mathbb{C})$  has an irreducible primitive action, but the corresponding singularity is neither exceptional nor weakly-exceptional (see [9, end of Section 3]).*

Since the  $A$ - $D$ - $E$  singularities are all quotient singularities, it makes sense to look at the case of quotient singularities in higher dimensions too.

**Remark 1.8.** *Consider the singularity  $\mathbb{C}^N/G$  (where  $G \subset GL_N(\mathbb{C})$ ). The definitions above mean that its exceptionality is dependent not on the group  $G$  itself, but on its image under the natural projection to  $PGL_N(\mathbb{C})$ . Thus, using the Chevalley–Shephard–Todd theorem (see [13, Theorem 4.2.5]), it is possible to simplify the problem by assuming that  $G \subset SL_N(\mathbb{C})$  (by assuming that  $G$  contains no pseudoreflections and then choosing a convenient lift of the image of  $G$  in  $PSL_N(\mathbb{C})$ ).*

A number of papers have been written about exceptional and weakly-exceptional quotient singularities in higher dimensions. The exceptional quotient singularities have all been classified in dimensions 3, 4, 5, 6 and 7 in [1], [2] and [7]. It has also been conjectured that there are no exceptional quotient singularities in higher dimensions.

The weakly-exceptional quotient singularities have also been classified in dimensions 3, 4 and 5 (see [9] and [10]). This paper generalises this classification to all other prime dimensions (i.e. dimensions  $q$ , where  $q$  is a prime integer) by the following result:

**Theorem 1.9** (Main theorem). *Let  $q$  be a positive prime integer. Then there are at most finitely many finite irreducible subgroups  $\Gamma \subset SL_q(\mathbb{C})$ , such that the singularity of  $\mathbb{C}^q/\Gamma$  is not weakly-exceptional.*

Note that the same result cannot hold in non-prime dimensions (for a counterexample, see [9, Theorem 1.15]). However, it is hoped that in non-prime dimensions all the counterexamples can be put into a small number of families — see Remark 3.12.

## 2. PRELIMINARIES

In order to prove the main theorem, several previously known results need to be considered. When trying to show that a quotient singularity is weakly-exceptional, the following results become very useful:

**Proposition 2.1.** *Let  $G \subset GL_N(\mathbb{C})$  be a finite subgroup, and  $H \subset G$  a subgroup. If the singularity of  $\mathbb{C}^N/G$  is not weakly-exceptional, then neither is the singularity of  $\mathbb{C}^N/H$ .*

*Proof:* Immediate from the definition.  $\square$

**Proposition 2.2.** *Let  $G \subset SL_N(\mathbb{C})$  be a finite subgroup with a semi-invariant of degree  $d < N$ . Then the singularity of  $\mathbb{C}^N/G$  is not weakly-exceptional.*

*Proof:* Immediate consequence of [1, Theorem 3.15].  $\square$

**Theorem 2.3** ([3, Theorem 1.12]). *Let  $G$  be a finite group in  $GL_{n+1}(\mathbb{C})$  that does not contain reflections. If  $\mathbb{C}^{n+1}/G$  is not weakly-exceptional, then there is a  $\bar{G}$ -invariant, irreducible, normal, Fano type projectively normal subvariety  $V \subset \mathbb{P}^n$  such that*

$$\deg V \leq \binom{n}{\dim V}$$

and for every  $i \geq 1$  and for every  $m \geq 0$  one has

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\dim V + 1) \otimes \mathcal{I}_V) = h^i(V, \mathcal{O}_V(m)) = 0,$$

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\dim V + 1) \otimes \mathcal{I}_V) \geq \binom{n}{\dim V + 1},$$

where  $\mathcal{I}_V$  is the ideal sheaf of the subvariety  $V \subset \mathbb{P}^n$ . Let  $\Pi$  be a general linear subspace of  $\mathbb{P}^n$  of codimension  $k \leq \dim V$ . Put  $X = V \cap \Pi$ . Then  $h^i(\Pi, \mathcal{O}_\Pi(m) \otimes \mathcal{I}_X) = 0$  for every  $i \geq 0$  and  $m \geq k$ , where  $\mathcal{I}_X$  is the ideal sheaf of the subvariety  $X \subset \Pi$ . Moreover, if  $k = 1$  and  $\dim V \geq 2$ , then  $X$  is irreducible, projectively normal, and  $h^i(X, \mathcal{O}_X(m)) = 0$  for every  $i \geq 1$  and  $m \geq 1$ .

Since these considerations rely on the group action used, one needs to define some terms common in the study of representation theory of finite groups:

**Definition 2.4.** *Given a group  $G \subset GL_N(\mathbb{C})$ , a system of imprimitivity for  $G$  is a set  $\{V_1, \dots, V_k\}$  of subspaces of  $\mathbb{C}^N$ , such that  $\dim V_i > 0 \forall i$ ,  $V_i \cap V_j = \{0\}$  whenever  $i \neq j$ ,  $V_1 \otimes \dots \otimes V_k = \mathbb{C}^N$ , and for any  $g \in G$  and  $i \in \{1, \dots, k\}$ , there exists  $j(g, i) \in \{1, \dots, k\}$ , such that  $g(V_i) = V_{j(g, i)}$ .*

**Remark 2.5.** *Clearly, any group  $G \subset GL_N(\mathbb{C})$  has at least one system of imprimitivity, namely,  $\{\mathbb{C}^N\}$ .*

**Definition 2.6.** *A group  $G \subset GL_N(\mathbb{C})$  is primitive if it has exactly one system of imprimitivity.*

This leads to the following well-known result:

**Lemma 2.7** (Jordan's theorem — see, for example, [5]). *For any given  $N$ , there are only finitely many finite primitive subgroups of  $SL_N(\mathbb{C})$ .*

**Definition 2.8.** *A group  $G \subset GL_N(\mathbb{C})$  is irreducible if for any system of imprimitivity  $\{V_1, \dots, V_k\}$  for  $G$ , the action of  $G$  permutes the subspaces  $V_1, \dots, V_k$  transitively.*

**Proposition 2.9.** *If a group  $G \subset GL_N(\mathbb{C})$  with a system of imprimitivity  $\{V_1, \dots, V_k\}$  is irreducible, then  $k$  divides  $N$ , and*

$$\dim V_1 = \dim V_2 = \dots = \dim V_k = N/k.$$

*Proof:* Since  $G$  is irreducible, given  $i, j \in \{1, \dots, k\}$ , there exists  $g_{i,j} \in G$  such that  $g_{i,j}(V_i) = V_j$ . Therefore,  $\dim V_i = \dim V_j$ . Applying this for different pairs  $(i, j)$ , get  $\dim V_1 = \dim V_2 = \dots = \dim V_k = d$ , some  $d \in \mathbb{Z}$ . Since the pairwise intersections between the  $V_i$ -s are trivial, and they span all of  $\mathbb{C}^N$ ,  $kd = N$ .  $\square$

**Definition 2.10.** A group  $G \subset GL_N(\mathbb{C})$  is monomial if there exists a system of imprimitivity  $\{V_1, \dots, V_k\}$  for  $G$ , such that for all  $i \in \{1, \dots, k\}$ ,  $\dim V_i = 1$ .

These groups have additional structure that can be exploited by using the following results:

**Proposition 2.11.** Let  $q > 1$  be a prime number, and let  $G \subset SL_q(\mathbb{C})$  be a finite irreducible subgroup. Then the action of  $G$  is either primitive or monomial.

*Proof:* Given any system of imprimitivity for  $G$ , Proposition 2.9 implies that all the subspaces in that system must have the same dimension  $d$ , with  $d|q$ . Since  $q$  is prime,  $d \in \{1, q\}$ . If there exists a system with 1-dimensional subspaces, then the action of  $G$  is monomial. Otherwise, the action of  $G$  must be primitive.  $\square$

**Proposition 2.12.** Let  $G \subset GL_N(\mathbb{C})$  be a finite monomial subgroup. Then have  $G \cong D \rtimes T$ , where  $D$  is abelian, and  $T \subseteq \mathbb{S}_N$ . Given a system of imprimitivity  $\{V_1, \dots, V_N\}$  for this group and choosing  $0 \neq x_i \in V_i$  for every  $i$ , the set  $\{x_1, \dots, x_N\}$  forms a basis for  $\mathbb{C}^N$ . In this basis, every element of  $D$  is a diagonal matrix, and for every element  $g \in G \setminus D$ , there exists some  $i, j \in \{1, \dots, N\}$  with  $i \neq j$  and  $g(x_i) \in V_j$ .

*Proof:* Since  $G$  is monomial, it has at least one system of imprimitivity  $\{V_1, \dots, V_k\}$ , such that all the  $V_i$ -s have dimension 1. Since  $V_1, \dots, V_k$  span  $\mathbb{C}^N$ , must have  $k = N$ . The action of  $G$  permutes  $V_1, \dots, V_N$ , so have a homomorphism  $\pi : G \rightarrow \mathbb{S}_N$  defined by these permutations. Let  $D = \ker(\pi) \trianglelefteq G$  and  $T = \text{Im}(\pi) \subseteq \mathbb{S}_N$ . Clearly,  $G = D \rtimes T$ .

For every  $i$ , pick a non-zero element  $x_i \in V_i$ . Since  $V_i$  is one-dimensional,  $x_i$  spans  $V_i$ , and so  $\{x_1, \dots, x_N\}$  is a basis for  $\mathbb{C}^N$ . Given any  $d \in D$ ,  $d(V_i) = V_i$  for every  $i$ , and so  $d$  must be a diagonal matrix in the chosen basis. Therefore,  $D$  is abelian.

Let  $g \in G$ , such that  $g(x_i) \in V_i$  for all  $i$ . Then  $\pi(g)$  is the trivial permutation in  $\mathbb{S}_N$ , and so  $g \in \ker(\pi) = D$ . So for any  $g \in G \setminus D$ , there exist  $i \neq j$  with  $g(x_i) \in V_j$ .  $\square$

**Proposition 2.13.** Let  $G \subset GL_N(\mathbb{C})$  be a finite monomial subgroup, and let  $G \cong D \rtimes T$  be the decomposition from Proposition 2.12. If  $G$  is irreducible, then  $T$  is transitive.

*Proof:* Assume  $T \subseteq \mathbb{S}_N$  is not transitive. Let  $x_1$  be a basis vector from Proposition 2.12. Consider the subspace  $V$  of  $\mathbb{C}^N$  spanned by  $\text{Orb}_G(x_1)$ . Since  $T$  is not transitive, there exists  $j \in \{1, \dots, N\}$  such that  $j \notin \text{Orb}_T(1)$ . Therefore,  $V_j \cap V = \{0\}$ , and so  $V \neq \mathbb{C}^N$ . However, by construction  $V$  must be  $G$ -invariant, and so  $G$  is not irreducible.  $\square$

**Lemma 2.14** ([11, §8.1]). If  $A$  is an abelian normal subgroup of a group  $G$ , then the degree of each irreducible representation of  $G$  divides the index  $(G : A)$  of  $A$  in  $G$ .

Finally, several miscellaneous results will be needed in the technical part of the proof:

**Theorem 2.15** ([6]). *Given  $m \in \mathbb{Z}$ ,  $m > 1$ , let  $W(m)$  be the set of integers  $n \geq 0$ , for which there exist  $\omega_1, \dots, \omega_n \in \mathbb{C}$  with  $\omega_i^m = 1 \ \forall i$  and  $\omega_1 + \dots + \omega_n = 0$ . Take  $m = p_1^{a_1} \dots p_r^{a_r}$  the prime decomposition of  $m$ . Then*

$$W(m) = \mathbb{N}p_1 + \mathbb{N}p_2 + \dots + \mathbb{N}p_r$$

**Definition 2.16** ([4]). *An  $n$ -by- $n$  matrix  $M$  is called circulant if it is of the form*

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_n & a_1 \end{pmatrix}$$

for some numbers  $a_1, \dots, a_n \in \mathbb{C}$ .

**Lemma 2.17** ([4, §3.2]). *For any circulant matrix  $M$  with  $a_1, \dots, a_n$  as above and any  $\omega \in \mathbb{C}$  with  $\omega^n = 1$ ,  $M$  has an eigenvector  $v = (1, \omega, \omega^2, \dots, \omega^{n-1})^T$  with eigenvalue  $\lambda = \sum_{i=1}^n a_i \omega^{i-1}$ . All the eigenvalues of  $M$  are of this form.*

*Proof:* It is easy to check that vectors of this form are indeed eigenvectors of  $M$  with relevant eigenvalues. These form a set of  $n$  linearly independent eigenvectors (can be seen via Theorem 2.15), so these are all the possible eigenvalues and eigenvectors for  $M$ .  $\square$

**Lemma 2.18.** *Let  $q \in \mathbb{N}$  be prime, and consider the following  $q \times q$  matrix with integer coefficients:*

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_{q-1} & a_q \\ a_q & a_1 & \cdots & a_{q-2} & a_{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_q & a_1 \end{pmatrix}$$

*Assume that  $a_i \geq 0 \ \forall i$ , and  $0 < d = \sum_{i=1}^q a_i < q$ . Then the determinant of  $M$  is not zero.*

*Proof:* Consider the matrix  $M$  over  $\mathbb{C}$ , and assume  $\det M = 0$ . Then one of the eigenvalues of  $M$  must be zero. So, by Lemma 2.17,

$$a_1 + \omega a_2 + \omega^2 a_3 + \dots + \omega^{q-1} a_q = 0$$

for some  $\omega$  with  $\omega^q = 1$ . Since all the  $a_i$ -s are non-negative integers, this is a sum of exactly  $d = \sum_{i=1}^q a_i$   $q$ -th roots of unity. So  $d \in W(q)$ . However, by Theorem 2.15 and using the fact that  $q$  is prime,  $W(q) = \mathbb{N}q$ . But by the initial assumption,  $0 < d < q$ , producing a contradiction.  $\square$

### 3. PROOF OF MAIN RESULT

The aim of this section is to prove Theorem 1.9. From now on, assume that  $q \geq 3$  is a prime, and  $\Gamma \subset \mathrm{SL}_q(\mathbb{C})$  be a finite irreducible subgroup, such that the singularity of  $\mathbb{C}^q/\Gamma$  is not weakly-exceptional.

By Jordan's theorem (see Lemma 2.7) there are only finitely many primitive finite subgroups of  $\mathrm{SL}_q(\mathbb{C})$ . Therefore, for the purposes of this proof, one can assume that the group  $\Gamma$  is imprimitive. Furthermore,  $q$  is assumed to be prime, so (by Proposition 2.11)  $\Gamma$  must be monomial.

**Lemma 3.1.** *Assume  $G \subset SL_q(\mathbb{C})$  is a finite irreducible monomial subgroup. Setting  $G \cong D \rtimes T$  as in Proposition 2.12, there exists  $\tau \in G \setminus D$  and a basis  $e_1, \dots, e_q$  for  $\mathbb{C}^q$ , such that  $\tau^q = \text{Id}_G$ , and  $\tau$  acts by*

$$\tau(e_i) = e_{i+1} \quad \forall i < q; \quad \tau(e_q) = e_1$$

*Proof:* Since  $G$  is irreducible,  $T$  must be a transitive subgroup of  $\mathbb{S}_q$  (by Proposition 2.13) and must thus contain a cycle of length  $q$  (since  $q$  is prime). Take  $\tau \in \Gamma$ , such that  $\pi(\tau)$  is a generator of this cycle. Let  $e_1 \in V_1$  be a non-zero vector. Then, renaming the  $V_i$ -s if necessary,  $\tau^i(e_1) \in V_{i+1}$  (for  $1 \leq i < q$ ). Set  $e_i = \tau^{i-1}(e_1)$  ( $2 \leq i \leq q$ ). Clearly,  $\tau(e_q) = \alpha e_1$  for some  $\alpha \in \mathbb{C}$ .

Since all the subspaces  $V_i$  are disjoint and one-dimensional,  $e_i$  must generate  $V_i$ , and so  $e_1, \dots, e_q$  must form a basis for  $\mathbb{C}^q$ . Also, since  $g \in D = \ker \pi$ , and  $\tau$  permutes the subspaces  $V_i$  non-trivially,  $\tau \notin D$ . Since  $\tau \in G \subseteq SL_q(\mathbb{C})$  and  $q$  odd, one also observes that  $\alpha = 1$ , and so  $\tau$  acts as stated above.  $\square$

**Corollary 3.2.** *There exists a subgroup  $G = D \rtimes \mathbb{Z}_q \subseteq \Gamma$  generated by  $D$  and  $\tau$ . The singularity of  $\mathbb{C}^q/G$  is not weakly-exceptional, and  $|\Gamma| \leq (q-1)!|G|$ .*

*Proof:* Take  $G$  generated by  $D$  and the element  $\tau \in \Gamma$  obtained in Lemma 3.1. Clearly,  $G \subseteq \Gamma$  and, looking at the action of  $\tau$ ,  $G \cong D \rtimes \mathbb{Z}_q$ . Since  $G \subseteq \Gamma$ , the singularity of  $\mathbb{C}^q/G$  is not weakly-exceptional by Proposition 2.1. Finally,  $\Gamma \subseteq D \rtimes \mathbb{S}_q$  (by Proposition 2.12), so

$$|\Gamma| \leq \frac{|\mathbb{S}_q|}{|\mathbb{Z}_q|} |G| = (q-1)!|G|$$

$\square$

From now on, fix the group  $G$  constructed above, the abelian normal subgroup  $D \triangleleft G$ , the element  $\tau \in G$  and the basis  $e_1, \dots, e_q$  for  $\mathbb{C}^q$  constructed in Lemma 3.1.

It is now advantageous to obtain a specialised criterion for determining whether or not a group of this form gives rise to a weakly-exceptional singularity.

**Proposition 3.3.** *Any irreducible representation of  $G$  (given above) over  $\mathbb{C}$  is either 1-dimensional or  $q$ -dimensional.*

*Proof:* Lemma 2.14 implies that  $(G : D) = q$ , which is a prime.  $\square$

**Lemma 3.4** (generalising [3, Theorem 3.4]). *Let  $q$  be an odd prime and assume  $G \subset SL_q(\mathbb{C})$  is a finite imprimitive subgroup isomorphic to  $A \rtimes \mathbb{Z}_q$  for some abelian  $A$ . Then the singularity of  $\mathbb{C}^q/G$  is not weakly-exceptional if and only if  $G$  has a (non-constant) semi-invariant of degree  $d < q$ .*

*Proof:* If  $G$  does have a semi-invariant of degree at most  $q-1$ , then the singularity is not weakly-exceptional by Proposition 2.2. Suppose that  $G$  does not have any such semi-invariants, but the singularity is not weakly-exceptional.

Then, by Theorem 2.3, there exists a  $\bar{G}$ -invariant irreducible normal Fano type variety  $V \subset \mathbb{P}^{q-1}$ , such that  $\deg V \leq \binom{q-1}{\dim V}$  and  $h^i(V, \mathcal{O}_V(m)) = 0 \quad \forall i \geq 1$   $\forall m \geq 0$  (where  $\mathcal{O}_V(m) = \mathcal{O}_V \otimes \mathcal{O}_{\mathbb{P}^{q-1}}(m)$ ).

Let  $n = \dim V$ . Then, since  $G$  has no semi-invariants of degree less than  $q$ , have  $n \leq q-2$ . Let  $\mathcal{I}_V$  be the ideal sheaf of  $V$ . Then

$$h^0(V, \mathcal{O}_V(m)) = h^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(m)) - h^0(\mathbb{P}^{q-1}, \mathcal{I}_V(m))$$

For instance,  $h^0(V, \mathcal{O}_V) = 1$ .

Take any  $m \in \mathbb{Z}$  with  $0 < m < q$ . Let  $W_m = H^0(\mathbb{P}^{q-1}, \mathcal{I}_V(m))$ . This is a linear representation of  $G$ , so  $q \mid \dim W_m$  (by Proposition 3.3, as  $G$  has no semi-invariants of degree  $m < q$ ). Since  $q \mid h^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(m))$ ,

$$h^0(V, \mathcal{O}_V(m)) \equiv 0 \pmod{q}$$

Since  $h^0(V, \mathcal{O}_V(t)) = \chi(V, \mathcal{O}_V(t))$  for any integer  $t \geq 0$ , there exist integers  $a_0, \dots, a_n$ , such that

$$h^0(V, \mathcal{O}_V(t)) = P(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$$

Consider  $P(t)$  as a polynomial over  $\mathbb{Z}_q$ . Since

$$P(m) = h^0(V, \mathcal{O}_V(m)) \equiv 0 \pmod{q}$$

whenever  $0 < m < q$ ,  $P(t)$  has at least  $q-1$  roots over  $\mathbb{Z}_q$ . But  $\deg P \leq n \leq q-2$ , so  $P(t)$  must be the zero polynomial over  $\mathbb{Z}_q$ . In particular,  $a_0 \equiv 0 \pmod{q}$ . On the other hand,  $a_0 = P(0) = h^0(V, \mathcal{O}_V) = 1 \not\equiv 0 \pmod{q}$ , leading to a contradiction.  $\square$

Now let  $f(x_1, \dots, x_q)$  be a semi-invariant of  $G$  of degree  $d < q$  as given in the above lemma. Using the chosen basis, let

$$m(x_1, \dots, x_q) = x_1^{a_1} x_2^{a_2} \dots x_q^{a_q}$$

be a monomial contained in  $f$  (for some  $a_i \in \mathbb{Z}_{\geq 0}$ , not all zero). This means that  $f$  must contain all the monomials in the  $\tau$ -orbit of  $m$ . Furthermore,  $\sum_i a_i = d < q$  and  $\sum_{i=0}^q \lambda^i \tau^i(m)$  is a semi-invariant of  $G$  whenever  $\lambda^q = 1$ . So, without loss of generality, can assume

$$f(x_1, \dots, x_q) = [m + \lambda \tau(m) + \dots + \lambda^{q-1} \tau^{q-1}(m)](x_1, \dots, x_q)$$

This semi-invariant can now be exploited to obtain a bound for the possible size of  $D$ .

First, consider the possible cyclic subgroups of  $D$ . Lemma 2.18 makes it possible to make the following deductions:

**Lemma 3.5.** *Let  $g \in D$ , and let  $n$  be the smallest positive integer, such that  $g^n$  is a scalar matrix. Then  $n < q^{2q+1}$ .*

*Proof:* Assume  $n > 1$ . Since  $g \in G \subset \mathrm{SL}_q(\mathbb{C})$ ,  $g^n = \zeta_q \mathrm{I}_q$ , where  $\zeta_q$  is a  $q$ -th root of 1 and  $\mathrm{I}_q$  is the identity matrix. Then, since all the elements of  $D$  are diagonal matrices,

$$g = \zeta_q^{\beta_0} \begin{pmatrix} \zeta_n^{\beta_1} & & \\ & \ddots & \\ & & \zeta_n^{\beta_q} \end{pmatrix}$$

where  $\beta_i \in \mathbb{Z}$ , not all zero, with  $0 \leq \beta_i < n \forall i > 0$ ;  $0 \leq \beta_0 < q$ . Since  $n$  was taken to be minimal, the highest common factor of  $\{n, \beta_1, \dots, \beta_q\}$  is 1.

Now consider the polynomial  $f$  of degree  $d < q$  described above. Since we know  $g \in G$ ,  $g(f) = \lambda f$  for some  $\lambda \in \mathbb{C}$ . Since  $g^{nq} = \mathrm{I}_q$  and all the monomials are

$g$ -semi-invariant,  $\lambda = \zeta_q^{\beta_0} \zeta_n^C$ , some  $C \in \mathbb{Z}$ . This is equivalent to:

$$\begin{aligned} C &\equiv a_1\beta_1 + a_2\beta_2 + \cdots + a_{q-1}\beta_{q-1} + a_q\beta_q \pmod{n} \\ &\equiv a_1\beta_2 + a_2\beta_3 + \cdots + a_{q-1}\beta_q + a_q\beta_1 \pmod{n} \\ &\equiv a_1\beta_3 + a_2\beta_4 + \cdots + a_{q-1}\beta_1 + a_q\beta_2 \pmod{n} \\ &\quad \dots \\ &\equiv a_1\beta_q + a_2\beta_1 + \cdots + a_{q-1}\beta_{q-2} + a_q\beta_{q-1} \pmod{n} \end{aligned}$$

This can be rewritten as

$$M(\beta_1, \dots, \beta_q)^T \equiv C(1, \dots, 1)^T \pmod{n}$$

where  $M$  is the matrix from Lemma 2.18). However, since  $\sum_{i=1}^q a_i = d$ ,  $M$  also satisfies

$$M(1, \dots, 1)^T = d(1, \dots, 1)^T$$

Take  $v = d(\beta_1, \dots, \beta_q)^T - C(1, \dots, 1)^T$ . By linearity,  $Mv \equiv 0 \pmod{n}$ . Multiplying both sides by the adjugate matrix of  $M$ , get:

$$\begin{aligned} (d\beta_1 - C) \det M &\equiv 0 \pmod{n} \\ (d\beta_2 - C) \det M &\equiv 0 \pmod{n} \\ &\quad \dots \\ (d\beta_q - C) \det M &\equiv 0 \pmod{n} \end{aligned}$$

Therefore,

$$d\beta_1 \det M \equiv d\beta_2 \det M \equiv \cdots \equiv d\beta_q \det M \equiv C \det M \pmod{n}$$

This implies that  $g^{d \det M}$  is a scalar matrix. By assumption,  $0 < d < q$  (in  $\mathbb{Z}$ ), and, by Lemma 2.18,  $\det M \neq 0$  (over  $\mathbb{Z}$ ), so  $|d \det M| = Kn$  for some positive integer  $K$ . Thus,  $n \leq |d \det M| \leq q |d \det M|$ .

Now look at the entries  $M_{i,j}$  of the matrix  $M$ . Since  $0 \leq a_k \leq d < q$  for all  $k$ ,  $|M_{i,j}| \leq d < q$ . Thus,

$$n \leq q |d \det M| \leq q \left( q \max_{i,j} |M_{i,j}| \right)^q < q^{2q+1}$$

□

**Corollary 3.6.** *Let  $\mathbb{Z}_m \subseteq D$ . Then  $m \leq q^{2q+2}$ .*

*Proof:* Take  $g$  a generator of  $\mathbb{Z}_m \subseteq D$ . Then for some  $n \leq q^{2q+1}$ ,  $g^n$  is a scalar matrix in  $\text{SL}_q(\mathbb{C})$ . Therefore,  $g^{qn} = \text{Id}_G$ . So

$$m \leq qn \leq q \cdot q^{2q+1} = q^{2q+2}$$

□

**Lemma 3.7.** *Let  $(\mathbb{Z}_m)^k \subseteq D \subset G \subseteq \Gamma \subset \text{SL}_q(\mathbb{C})$ . Then  $k \leq q$ .*

*Proof:* Let  $g_1, \dots, g_k$  be a minimal set of generators of  $(\mathbb{Z}_m)^k \subseteq D$ . Then for every  $i > 1$ ,  $g_i \notin \langle g_1, \dots, g_{i-1} \rangle$ . Let  $\zeta_m$  be a primitive  $m$ -th root of 1. Then all the  $g_i$  are diagonal matrices with some powers of  $\zeta_m$  as diagonal entries. But any matrix in  $\text{SL}_q(\mathbb{C})$  has exactly  $q$  diagonal entries, so at most  $q$  such  $g_i$ -s can be chosen. Therefore,  $k \leq q$ . □

**Corollary 3.8.**  $D \subseteq \bigotimes_{i=0}^{q^{2q+2}} (\mathbb{Z}_i)^q$ .



*Proof:* Immediate from Corollary 3.6 and Lemma 3.7.  $\square$

**Theorem 3.9.** *Given  $q > 3$ , there are at most finitely many finite irreducible monomial groups  $\Gamma \subseteq SL_q(\mathbb{C})$ , such that the singularity of  $\mathbb{C}^q/\Gamma$  is not weakly-exceptional.*

*Proof:* Let  $\Gamma$  be such a group. Then by Corollary 3.2, there exists  $G \subseteq \Gamma$ , such that  $G \cong D \rtimes \mathbb{Z}_q$  and  $|\Gamma| \leq (q-1)!|G|$ . By Corollary 3.8,  $D \subseteq \bigotimes_{i=0}^{q^{2q+2}} (\mathbb{Z}_i)^q$ , so there are at most finitely many such group  $D$ . It follows that there are at most finitely many such groups  $G$ , and hence at most finitely many such groups  $\Gamma$ .  $\square$

**Remark 3.10.** *The bounds used here for the possible sizes of the groups  $D$ ,  $G$  and  $\Gamma$  are by no means effective. However, improving them would make the proofs in this section a lot more technically complicated, and would not provide much insight into the structure of these groups or significantly improve the main result of this paper.*

This result provides the last step needed for the proof of the main theorem of this paper:

*Proof of Theorem 1.9:* Let  $q$  be any positive prime integer. If  $q = 2$ , no such groups  $\Gamma$  exists (by Example 1.4), so assume  $q \geq 3$ . Then, by Proposition 2.11,  $\Gamma$  is either monomial or primitive. If  $\Gamma$  is primitive, then it must be among a finite list of groups by Lemma 2.7. If  $\Gamma$  is monomial, it must belong to a finite set of groups by Theorem 3.9. Therefore, there are only finitely many such subgroups  $\Gamma \subset SL_q(\mathbb{C})$ .  $\square$

Note that the proof of this theorem also provides the means of enumerating all the imprimitive groups that  $\Gamma$  can be isomorphic (or conjugate) to for any given  $q$ . This would rely on making a list of all the possible matrices  $M$  from Lemma 3.5 and computing (the prime factorisations of) their determinants.

Although the bounds seen in the proofs are very large, choosing a specific value of  $q$  quickly produces a fairly short list for the possible isomorphism classes of  $D$ :

**Example 3.11.** *If  $q = 7$ , then  $D \subseteq \mathbb{Z}_7 \times (\mathbb{Z}_{n \cdot d})^6$ , where the values of  $n$  and  $d$  are as follows:*

$$\begin{aligned} d = 2 \quad n &= 2^6 \\ d = 3 \quad n &\text{ is one of } 2^3, 3^6, 43 \\ d = 4 \quad n &\text{ is one of } 2^{12}, 29, 71, 547 \\ d = 5 \quad n &\text{ is one of } 2^6, 2^3 \cdot 71, 5^6, 13^2, 29 \cdot 113, 43, 197, 421, 463 \\ d = 6 \quad n &\text{ is one of } 2^9, 2^6 \cdot 3^6, 2^3 \cdot 29, 2^6 \cdot 43, 13^2, 29^2, 29 \cdot 449, 41^2, 43 \cdot 71, \\ &\quad 113, 197, 211, 379, 463, 757, 2689. \end{aligned}$$

*Proof:* By easy direct computation.  $\square$

A further computation (to obtain the possible actions of the group  $T$  on  $D \subset D \rtimes T = \Gamma$ ) can reduce this list even further. An example of such a computation can be seen for the case of  $q = 5$  in [9].

**Remark 3.12.** *It is easy to see that there is no hope to have the same result when the dimension  $N$  is not a prime number. The easiest way to see counterexamples for arbitrarily high dimension is to take  $N = n^2$  and write the coordinates of  $\mathbb{C}^N$  as entries of an  $n \times n$  matrix. This gives a map  $\iota : SL_n(\mathbb{C}) \times SL_n(\mathbb{C}) \rightarrow SL_N(\mathbb{C})$ , where the copies of  $SL_n(\mathbb{C})$  act by left and (transposed) right multiplication. It is easy to choose pairs of finite subgroups  $A, B \subset SL_n(\mathbb{C})$ , such that  $\Gamma = \iota(A, B)$*

acts irreducibly, and the action of  $\Gamma$  clearly has a degree  $n$  invariant (given by the matrix determinant). Clearly, one can choose infinitely many different suitable pairs  $(A, B)$ , yielding infinitely many such groups  $\Gamma \subset SL_N(\mathbb{C})$ .

It is, however, hoped that, for any given dimension  $N$ , groups in the image of  $SL_a(\mathbb{C}) \times SL_b(\mathbb{C}) \rightarrow SL_{ab}(\mathbb{C})$  account for all the infinite families of groups that give rise to singularities that are not weakly-exceptional.

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